Sec 3.4 Matrix Operations

matrices and vectors always underlined!

$\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ entries (aka "elements") are a_{ij} ← row #, ← column #

a_{21} is the "(2,1) – entry"

Scalar multiplication: $3\underline{A} = \begin{bmatrix} 3a_{11} & 3a_{12} \\ 3a_{21} & 3a_{22} \end{bmatrix}$

(a number)

Addition: • Done "elementwise".

– Dimensions (#rows and # columns) must match (between matrices)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a+\alpha & b+\beta \\ c+\gamma & d+\delta \end{bmatrix}$$

Matrix multiplication ★ NOT done elementwise

– Requirement: For product $\underline{A}\,\underline{B}$ to be defined, #(cols of $\underline{A}$) must match #(rows of $\underline{B}$)

– First idea: $\underline{a} = [a_1 \ a_2 \ a_3]_{(1\times 3)}$ $\quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{(3\times 1)}$

$\underline{a}\,\underline{x}$: $(1\times \underbrace{3)(3}_{\text{match ✓}}\times 1)$ → product defined

result will be a (1×1) matrix ... but a (1×1) matrix is just a number / scalar.

$$\underline{a}\,\underline{x} = [\underbrace{a_1 \ a_2 \ a_3}_{\text{pairing}}] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \overset{\text{DEF}}{=} x_1 a_1 + x_2 a_2 + x_3 a_3$$

(like dot product)

(pairing is why we needed match)

Let

$$\underline{v_1} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \qquad \underline{v_2} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \qquad \underline{v_3} = \begin{bmatrix} 7 \\ 1 \\ -3 \end{bmatrix}$$

all 3×1 (column) vectors.

$$(3\times 3) \quad \underline{A} = \begin{bmatrix} \underline{v_1} & \underline{v_2} & \underline{v_3} \end{bmatrix} \overset{\text{DEF}}{=} \begin{bmatrix} 1 & 2 & 7 \\ 3 & 0 & 1 \\ 5 & -1 & -3 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \left(\text{equiv to } \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} \right)$$

$\underline{Ax}$ defined? $(3\times \underbrace{3)(3}_{\checkmark}\times 1) \longrightarrow$ result (3×1)

($\underline{xA}$ not defined because of mismatch)

$$\star \quad \underline{Ax} = \begin{bmatrix} \underline{v_1} & \underline{v_2} & \underline{v_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \overset{\text{DEF}}{=} x_1\underline{v_1} + x_2\underline{v_2} + x_3\underline{v_3}$$

(x_i are "weights" for $\underline{v_i}$)

Ex

$$\underbrace{\begin{bmatrix} 1 & 2 & 7 \\ 3 & 0 & 1 \\ 5 & -1 & -3 \end{bmatrix}}_{[\underline{v_1}\ \underline{v_2}\ \underline{v_3}] = \underline{A}} \underbrace{\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}}_{\underline{x}} = 3\underline{v_1} + \underline{v_2} - \underline{v_3}$$

$$= \begin{bmatrix} 3 \\ 9 \\ 15 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -7 \\ -1 \\ 3 \end{bmatrix}$$

$$\underline{Ax} = \begin{bmatrix} -2 \\ 8 \\ 17 \end{bmatrix}$$

Now keep $\underline{A}$, $\underline{x}$, let $\underline{y} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$,

$$\underline{B} = \begin{bmatrix} \underline{x} & \underline{y} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \\ -1 & 3 \end{bmatrix}_{(3\times 2)}$$

$\underline{AB}$ defined? $(3\times \underbrace{3)(3}_{\checkmark}\times 2)$ result $\longrightarrow (3\times 2)$

DEF $\underline{AB} = \underline{A}[\underline{x}\ \ \underline{y}] \overset{\text{DEF}}{=} [\underbrace{\underline{Ax}}_{3\times 1}\ \ \underbrace{\underline{Ay}}_{3\times 1}]$

$$\underline{Ex}: \quad \underline{AB} = \underbrace{\begin{bmatrix} 1 & 2 & 7 \\ 3 & 0 & 1 \\ 5 & -1 & -3 \end{bmatrix}}_{\underline{A}} \begin{bmatrix} 3 & 2 \\ 1 & 1 \\ -1 & 3 \end{bmatrix}$$

$$\begin{array}{ccccc} \underline{v_1} & \underline{v_2} & \underline{v_3} & \underline{x} & \underline{y} \end{array}$$

$$\underline{Ax} = \begin{bmatrix} -2 \\ 8 \\ 17 \end{bmatrix}$$

$$\underline{Ay} = 2\underline{v_1} + \underline{v_2} + 3\underline{v_3} = \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 21 \\ 3 \\ -9 \end{bmatrix}$$

$$= \begin{bmatrix} 25 \\ 9 \\ 0 \end{bmatrix}$$

$$\text{so} \quad \underline{AB} = [\underline{Ax}\ \ \underline{Ay}] = \begin{bmatrix} -2 & 25 \\ 8 & 9 \\ 17 & 0 \end{bmatrix}$$

$\underline{BA}: (3\times \underbrace{2)(3}_{\times}\times 3)$ so $\underline{BA}$ undefined.

Even if $\underline{BA}$ defined, $\boxed{\text{often } \underline{BA} \neq \underline{AB}}$

"<u>vector equations</u>"

$$\underline{A} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$$

$$= [\underline{u}\ \ \underline{v}]$$

$$\underline{Ax} = \underline{b} \iff \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$$

$$\iff x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$$

$$\iff \begin{bmatrix} x_1 + 3x_2 \\ 2x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$$

same as $\begin{cases} x_1 + 3x_2 = 9 \\ 2x_1 + x_2 = 8 \end{cases}$

solution is $\begin{matrix} x_1 = 3 \\ x_2 = 2 \end{matrix}$

target $\underline{b} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$

$b = 3\underline{u} + 2\underline{v}$